Tentamen Complexe Analyse 09/02/07

1. $P(z) = \sum_{k=0}^{n} a_k z^k$ $\quad a_k$ reëel $\Rightarrow$ $\overline{P(z)} = \overline{\sum_{k=0}^{n} a_k z^k} = \sum_{k=0}^{n} a_k \bar{z}^k = P(\bar{z})$

dus $P(z)=0 \Leftrightarrow P(\bar{z})=0$

2. $f(z) = \frac{z^2}{z+2}$ $\quad$ Voor $z=0$ is $\frac{1}{z+2} \neq 0 \Rightarrow f(z)$ anal. in $z=0$; quotient van anal. functies

$$f(z) = z^2 \frac{1}{2[1+\frac{z}{2}]} = \frac{1}{2}z^2\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{2^3}+\cdots\right) = \frac{z^2}{2}-\frac{z^3}{4}+\frac{z^4}{8}-\frac{z^5}{16}+\cdots$$

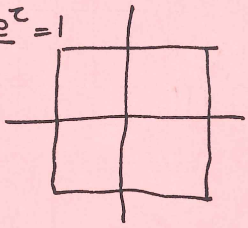
convergentie straal $R=2$ door meetkundige reeks òf eerste sing. vanuit 0 gezien

3. $f(z) = \frac{\sin z}{z^3}e^z = \frac{1}{z^3}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]\left[1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right]$

$$= \frac{1}{z^3}\left[z+z^2+z^3\left(\frac{1}{2!}-\frac{1}{3!}\right)+\cdots\right] = \frac{1}{z^3}\left[z+z^2+\frac{1}{3}z^3+\cdots\right]$$

$= \frac{1}{z^2}+\frac{1}{z}+\frac{1}{3}+\cdots$ $\quad$ residue $f(z)$ in $z=0$: 1

Rechtstreeks: $\lim_{z\to 0}\frac{d}{dz}\left[z^2 \frac{\sin z}{z^3}e^z\right] = \lim_{z\to 0}\frac{z(\cos z+\sin z)e^z-\sin z\, e^z}{z^2}=1$



4. $f(z)=e^{iz^2}$ $\quad z=x+iy$ $\quad z^2=x^2-y^2+2ixy$ $\quad |f(z)| = e^{-2xy}$

$x=1$ $\quad \max|f(z)| = \max_{-1\le y\le 1} e^{-2y} = e^2$ $\quad$ symmetry

$\max|f(z)|=e^2$ $\quad$ NB. $|f(z)|$ continu op $|z|\leq 1$ dus max bestaat (Weierstrass)
$f(z)$ anal. op $|z|\leq 1$ dus $\max|f(z)|$ op rand

5. $\int_0^{2\pi}\frac{\cos\theta}{2-\sin\theta}d\theta$ $\quad z=e^{i\theta}$ $\quad dz=ie^{i\theta}d\theta$ $\quad d\theta=\frac{dz}{iz}$

$$= \int_{|z|=1}\frac{(z+1/z)/2}{2-(z-1/z)/2i}\frac{dz}{iz} = \int_{|z|=1}\frac{z^2+1}{4z-(z^2-1)/i}\frac{dz}{iz} = \int_{|z|=1}\frac{z^2+1}{-z^2+4iz+1}\frac{dz}{z}$$

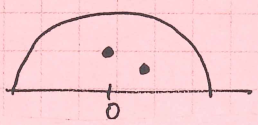
$= -\int_{|z|=1}\frac{z^2+1}{z^2-4iz-1}\frac{dz}{z}$ $\quad z^2-4iz-1 = z^2-4iz-4+4-1 = (z-2i)^2+3$
$z = 2i\pm\sqrt{3}i = (2\pm\sqrt{3})i$

$= -2\pi i\left[\text{Res}_{z=0}+\text{Res}_{(2-\sqrt{3})i}\right] = -2\pi i\left[-1+1\right]=0$ $\quad$ NB $(z+\frac{1}{z})\frac{1}{2z-4i}=1$ als $z=(2-\sqrt{3})i$

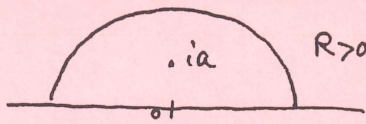
"Rechtstreeks": $\int_0^{2\pi}\frac{\cos\theta}{2-\sin\theta}d\theta = \int_{-\pi}^{\pi}\frac{\cos\theta}{2-\sin\theta}d\theta = \int_{\mathbb{R}}\frac{1-t^2}{1+t^2}\frac{1}{t^2-t+1}dt$ $\quad \left\{\begin{array}{l} t=\tan\frac{1}{2}\theta \\ \cos\theta = \frac{1-t^2}{1+t^2} \quad d\theta=\frac{2}{1+t^2}dt \\ \sin\theta=\frac{2t}{1+t^2}\end{array}\right.$

$\Rightarrow \int_C \frac{1-t^2}{1+t^2}\frac{1}{t^2-t+1}dt = 2\pi i\left[\text{Res}_{t=i}+\text{Res}_{t=\frac{1}{2}+i\frac{1}{2}\sqrt{3}}\right] = 2\pi i\left[\frac{2}{2i}\cdot\frac{1}{-i}+\frac{\frac{3}{2}-\frac{1}{2}i\sqrt{3}}{\frac{1}{2}+\frac{1}{2}i\sqrt{3}}\frac{1}{i\sqrt{3}}\right]$

$= 2\pi i[1-1]=0 \Rightarrow \int_{\mathbb{R}}\frac{1-t^2}{1+t^2}\frac{1}{t^2-t+1}dt=0$



6. $\int_{\mathbb{R}}\frac{x\sin x}{x^2+a^2}dx$ $\qquad \int_C\frac{ze^{iz}}{z^2+a^2}dz$



$R>a$

$\int_C\frac{ze^{iz}}{z^2+a^2}dz = 2\pi i\,\text{Res}_{z=ia}\frac{ze^{iz}}{z^2+a^2} = 2\pi i\cdot\frac{e^{-a}}{2} = \pi i e^{-a}$

$\int_C\frac{ze^{iz}}{z^2+a^2}dz = \int_{-R}^{R}\frac{xe^{ix}}{x^2+a^2}dx + \int_{C_R}\frac{ze^{iz}}{z^2+a^2}dz$ $\quad \left|\int_{C_R}\right| = \int_{C_R}|e^{iz}||dz| \to 0$

$\Rightarrow \int_{\mathbb{R}}\frac{x\cos x+ix\sin x}{x^2+a^2}dx = \pi i e^{-a} \Rightarrow \int_{\mathbb{R}}\frac{x\cos x}{x^2+a^2}dx=0$ $\quad \int_{\mathbb{R}}\frac{x\sin x}{x^2+a^2}dx=\pi e^{-a}$